ON QUOTIENT MODULES OF $H^2(\mathbb{D}^n)$: ESSENTIAL NORMALITY AND BOUNDARY REPRESENTATIONS

B. KRISHNA DAS, SUSHIL GORAI, AND JAYDEB SARKAR

ABSTRACT. Let $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$, $n \geq 1$, and let $H^2(\mathbb{D}^n)$ be the Hardy space over $\mathbb{D}^n$. For $n \geq 3$, we show that if $\theta \in H^\infty(\mathbb{D}^n)$ is an inner function, then the n-tuple of commuting operators $(C_{z_1}, \ldots, C_{z_n})$ on the Beurling type quotient module $\mathcal{Q}_\theta$ is not essentially normal, where

$$\mathcal{Q}_\theta = H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n) \quad \text{and} \quad C_{z_j} = P_{\mathcal{Q}_\theta} M_{z_j}|_{\mathcal{Q}_\theta} \quad (j = 1, \ldots, n).$$

Rudin's quotient modules of $H^2(\mathbb{D}^2)$ are also shown to be not essentially normal. We prove several results concerning boundary representations of C^*-algebras corresponding to different classes of quotient modules including doubly commuting quotient modules, homogenous quotient modules and Clark type quotient modules.

1. INTRODUCTION

Let $H^2(\mathbb{D}^n)$, $n \geq 1$, denote the Hardy space of holomorphic functions on the unit polydisc $\mathbb{D}^n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| \leq 1, i = 1, \ldots, n\}$, that is,

$$H^2(\mathbb{D}^n) = \Big\{ f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}} \in \mathcal{O}(\mathbb{D}^n) : \|f\|^2 := \sum_{\boldsymbol{k} \in \mathbb{N}^n} |a_{\boldsymbol{k}}|^2 < \infty \Big\},$$

where $\mathbb{N}$ is the set of all natural numbers including 0, $\mathbb{N}^n = \{\boldsymbol{k} = (k_1, \ldots, k_n) : k_j \in \mathbb{N}, j = 1, \ldots, n\}$ and $\boldsymbol{z}^{\boldsymbol{k}} := z_1^{k_1} \cdots z_n^{k_n}$. It is well known that $H^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel

$$\mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n (1 - z_i \bar{w}_i)^{-1}, \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

and $(M_{z_1}, \ldots, M_{z_n})$ is a commuting tuple of isometries on $H^2(\mathbb{D}^n)$, where

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}) \qquad (f \in H^2(\mathbb{D}^n), \boldsymbol{w} \in \mathbb{D}^n, i = 1, \ldots, n).$$

We represent the n-tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$ as a Hilbert module over $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_n]$ with the following module action:

$$\mathbb{C}[\boldsymbol{z}] \times H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n), \quad (p, f) \mapsto p(M_{z_1}, \ldots, M_{z_n}) f.$$

With the above module action $H^2(\mathbb{D}^n)$ is called the Hardy module over $\mathbb{C}[\boldsymbol{z}]$. A closed subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is called a *submodule* if $M_{z_i} \mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$, and a closed

2000 *Mathematics Subject Classification.* 47A13, 47A20, 47L25, 47L40, 46L05, 46L06.

Key words and phrases. Hardy space, reproducing kernel Hilbert spaces, quotient modules, essential normality, boundary representation.

subspace $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ is a *quotient module* if $\mathcal{Q}^\perp$ ($\cong H^2(\mathbb{D}^n)/\mathcal{Q}$) is a submodule. A quotient module $\mathcal{Q}$ is said to be of *Beurling type* (cf. [12]) if

$$\mathcal{Q} = \mathcal{Q}_\theta := H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n) \cong H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$$

for some inner function $\theta \in H^\infty(\mathbb{D}^n)$ (that is, θ is a bounded analytic function on $\mathbb{D}^n$ and $|\theta| = 1$ a.e. on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$). We denote by $\mathcal{S}_\theta$ the submodule $\theta H^2(\mathbb{D}^n)$ of $H^2(\mathbb{D}^n)$.

A quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ is *essentially normal* (cf. [6]) if the commutator $[C_{z_i}^{\mathcal{Q}}, C_{z_j}^{\mathcal{Q}*}]$ is compact for all $1 \le i, j \le n$, where $C_{z_i}^{\mathcal{Q}}$ is the compression of the shift M_{z_i} to $\mathcal{Q}$, that is,

$$C_{z_i}^{\mathcal{Q}} := P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} \qquad (i = 1, \ldots, n).$$

We use the notation C_{z_i} when $\mathcal{Q}$ is clear from the context. We also denote by $B(\mathcal{Q})$ and $C^*(\mathcal{Q})$ the Banach algebra and the C^*-algebra generated by $\{I_{\mathcal{Q}}, C_{z_i}\}_{i=1}^n$ respectively. For convenience in notation we put

$$B(\mathcal{Q}) = B(C_{z_1}, \ldots, C_{z_n}), \text{ and } C^*(\mathcal{Q}) = C^*(C_{z_1}, \ldots, C_{z_n}).$$

It is well known that for an essentially normal quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$, $B(\mathcal{Q})$ is an irreducible operator algebra and the C^*-algebra $C^*(\mathcal{Q})$ contains all compact operators in $\mathcal{B}(\mathcal{Q})$.

Essential normality of Hilbert modules is a much studied object in operator theory and function theory. It establishes important connections between operator theory, algebraic geometry, homology theory and complex analysis through the BDF theory [3]. It is well known that any proper quotient module of $H^2(\mathbb{D})$ is of Beurling-type and essentially normal. This, however, does not hold in general:

(1) For $n = 2$ a Beurling type quotient module $\mathcal{Q}_\theta \subseteq H^2(\mathbb{D}^2)$ is essentially normal if and only if θ is a rational inner function of degree at most $(1, 1)$ [12].

(2) For $n \ge 2$, a quotient module $\mathcal{Q}$ is a Beurling type quotient module of $H^2(\mathbb{D}^n)$ if and only if $\mathcal{Q}^\perp$ is a doubly commuting submodule [18].

An incomplete list of references on the study of essential normality of different classes quotient modules, including Clark type quotient modules and homogeneous quotient modules, over bidisc is: [6], [7], [11], [12], [13] and [19].

In this paper we first investigate the essential normality of certain classes of quotient modules including Beurling-type quotient modules of $H^2(\mathbb{D}^n)$, $n \ge 3$. We prove that the Beurling type quotient modules of $H^2(\mathbb{D}^n)$ ($n \ge 3$) and Rudin quotient modules of $H^2(\mathbb{D}^2)$ are not essentially normal. We obtain a complete characterization for essential normality of doubly commuting quotient modules of an analytic Hilbert module (defined in Section 2) over $\mathbb{C}[\boldsymbol{z}]$ including $H^2(\mathbb{D}^n)$ and the weighted Bergman modules $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$ ($\boldsymbol{\alpha} \in \mathbb{Z}^n, \alpha_i > -1, i = 1, \ldots, n$) as special cases ($n \ge 2$).

We also study boundary representations, in the sense of Arveson ([1], [2]), of the C^*-algebra $C^*(\mathcal{Q})$ for different classes of quotient modules $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$. Before we describe them, let us recall the notion of boundary representations and some relevant results. Let A be an operator algebra with identity, and let $C^*(A)$ be the C^*-algebra generated by A. An irreducible representation ω of $C^*(A)$ is a *boundary representation* relative to A if $\omega|_A$ has a unique completely positive (CP) extension to $C^*(A)$. An operator algebra A has sufficiently many boundary representations if

$$\bigcap_{\omega\in \mathrm{bdy}(A)} \ker\omega = \{0\},$$

where bdy(A) denotes the set of all boundary representations of $C^*(A)$ relative to A. It is worth mentioning here that, by a recent work of Davidson and Kenedy ([9]), the Silov boundary ideal of A (in the sense of Arveson [1]) is $\cap_{\omega\in \mathrm{bdy}(A)}\ker\omega$. It is of great interest and importance to identify operator algebras with sufficiently many boundary representations. In the particular case of irreducible operator algebras containing compact operators, the existence of sufficiently many boundary representations and the fact that the identity representation is a boundary representation are closely related.

Theorem 1.1 ([2]). *Let A be an irreducible operator algebra with identity, and let $C^*(A)$ contain all the compact operators. Then A has sufficiently many boundary representations if and only if the identity representation of $C^*(A)$ is a boundary representation relative to A.*

In our context, if $\mathcal{Q}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$ then $B(\mathcal{Q})$ is irreducible and $C^*(\mathcal{Q})$ contains all the compact operators on $\mathcal{Q}$. Therefore, it is natural to ask whether the identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$ for the case when $\mathcal{Q}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$. This problem has a complete solution for the case $n=1$ (see Arveson [1],[2]):

Theorem 1.2 (Arveson). *Let $\mathcal{Q}_\theta$ be a quotient module of $H^2(\mathbb{D})$. Then the identity representation of $C^*(\mathcal{Q}_\theta)$ is a boundary representation relative to $B(\mathcal{Q}_\theta)$ if and only if Z_θ is a proper subset of $\mathbb{T}$, where Z_θ consists of all points λ on $\mathbb{T}$ for which θ cannot be continued analytically from $\mathbb{D}$ to λ.*

For the class of essentially normal Beurling type quotient modules of $H^2(\mathbb{D}^2)$, the following characterization was obtained in [12].

Theorem 1.3 (Guo and Wang). *Let $\theta\in H^\infty(\mathbb{D}^2)$ be a rational inner function of degree at most $(1,1)$, and $\mathcal{Q}_\theta$ be the corresponding essentially normal quotient module of $H^2(\mathbb{D}^2)$. Then the identity representation of $C^*(\mathcal{Q}_\theta)$ is a boundary representation relative to $B(\mathcal{Q}_\theta)$ if and only if θ is not a one variable Blaschke factor.*

In this paper, we study the same problem for several classes of quotient modules of some Hilbert modules over $\mathbb{D}^n$, $n\geq 2$. To be more precise, we study boundary representations for doubly commuting quotient modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, and obtain some direct results for the case of $H^2(\mathbb{D}^n)$ and $L^2_a(\mathbb{D}^n)$ $(n\geq 2)$ (see Theorems 4.4 and 4.5). We also considered the class of Clark type (denoted by $\mathcal{Q}_{\boldsymbol{\eta}}$) quotient modules of $H^2(\mathbb{D}^n)$ and homogeneous quotient modules of $H^2(\mathbb{D}^2)$.

The paper is organized as follows. After obtaining some preliminary results in Section 2, we consider essential normality of Beurling type quotient module of $H^2(\mathbb{D}^n)$ $(n\geq 3)$, doubly commuting quotients modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ and Rudin quotient module of $H^2(\mathbb{D}^2)$ in Section 3. Section 4 is devoted to the study of boundary representations for doubly commuting quotient modules. In Section 5 and 6, we discuss boundary representations for Clark type (denoted by $\mathcal{Q}_{\boldsymbol{\eta}}$) quotient modules of $H^2(\mathbb{D}^n)$ and homogeneous quotient modules of $H^2(\mathbb{D}^2)$, respectively.

2. Preparatory results

In this section we recall some definitions, and prove some elementary results which will be used later. For each $\boldsymbol{w} \in \mathbb{D}^n$, the *normalized kernel* function $K_{\boldsymbol{w}}$ of $H^2(\mathbb{D}^n)$ is defined by

$$K_{\boldsymbol{w}}(\boldsymbol{z}) := \frac{1}{\|\mathbb{S}(\cdot, \boldsymbol{w})\|}\mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} \sqrt{(1-|w_i|^2)}\frac{1}{1-\overline{w}_i z_i} \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

where $\mathbb{S}(\cdot, \boldsymbol{w})(\boldsymbol{z}) = \mathbb{S}(\boldsymbol{z}, \boldsymbol{w})$ for all $\boldsymbol{z} \in \mathbb{D}^n$.

Lemma 2.1. *Let $l \in \{1, \ldots, n\}$ be a fixed integer, and let $\boldsymbol{w}_l = (w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_n)$ be a fixed point in $\mathbb{D}^{n-1}$. Then $K_{(\boldsymbol{w}_l, w)}$ converges weakly to 0 as w approaches to $\partial\mathbb{D}$, where $(\boldsymbol{w}_l, w) = (w_1, \ldots, w_{l-1}, w, w_{l+1}, \ldots, w_n)$.*

Proof. For each $p \in \mathbb{C}[\boldsymbol{z}]$,

$$\langle K_{(\boldsymbol{w}_l, w)}, p\rangle = \overline{p(\boldsymbol{w}_l, w)}\sqrt{1-|w|^2} \prod_{i=1, i\neq l}^{n} \sqrt{1-|w_i|^2}, \tag{2.1}$$

which converges to zero as w approaches to $\partial\mathbb{D}$. For an arbitrary $f \in H^2(\mathbb{D}^n)$, the result now follows from the fact that $\|K_{\boldsymbol{\lambda}}\| = 1$ for all $\boldsymbol{\lambda} \in \mathbb{D}^n$ and $\mathbb{C}[\boldsymbol{z}]$ is dense in $H^2(\mathbb{D}^n)$. $\square$

For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$. For an inner function $\theta \in H^\infty(\mathbb{D}^n)$, it is well known that

$$P_{\mathcal{S}_\theta} = M_\theta M_\theta^* \quad \text{and} \quad P_{\mathcal{Q}_\theta} = I_{H^2(\mathbb{D}^n)} - M_\theta M_\theta^*,$$

where M_θ is the multiplication operator defined by

$$(M_\theta f)(\boldsymbol{w}) = \theta(\boldsymbol{w})f(\boldsymbol{w}) \qquad (\boldsymbol{w} \in \mathbb{D}^n, f \in H^2(\mathbb{D}^n)).$$

It follows from the reproducing property of the Szegö kernel that

$$M_\theta^* K(\cdot, \boldsymbol{w}) = \overline{\theta(\boldsymbol{w})}K(\cdot, \boldsymbol{w}),$$

where $K(\cdot, \boldsymbol{w}) := K_{\boldsymbol{w}}$, $\boldsymbol{w} \in \mathbb{D}^n$. In particular, one has

$$P_{\mathcal{S}_\theta}(K_{\boldsymbol{w}}) = M_\theta M_\theta^* K_{\boldsymbol{w}} = \overline{\theta(\boldsymbol{w})}\theta K_{\boldsymbol{w}} \qquad (\boldsymbol{w} \in \mathbb{D}^n).$$

These observations yield the following lemma.

Lemma 2.2. *Let θ be an inner function in $H^\infty(\mathbb{D}^n)$. Then*

$$P_{\mathcal{Q}_\theta}(K_{\boldsymbol{w}}) = (1 - \overline{\theta(\boldsymbol{w})}\theta)K_{\boldsymbol{w}} \qquad (\boldsymbol{w} \in \mathbb{D}^n). \tag{2.2}$$

We now recall the definition of an analytic Hilbert module over $\mathbb{C}[z]$ (see [4]). Let $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be a positive definite function such that $k(z, w)$ is analytic in z and anti-analytic in w. Let $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}, \mathbb{C})$ be the corresponding reproducing kernel Hilbert space. The Hilbert space $\mathcal{H}_k$ is said to be *reproducing kernel Hilbert module* over $\mathbb{C}[z]$ if the multiplication operator M_z is bounded on $\mathcal{H}_k$.

Definition 2.3. A reproducing kernel Hilbert module $\mathcal{H}_k$ over $\mathbb{C}[z]$ is said to be an analytic Hilbert module over $\mathbb{C}[z]$ if $k^{-1}(z,w)$ is a polynomial in z and $\bar{w}$.

Typical examples of analytic Hilbert modules are the Hardy module $H^2(\mathbb{D})$ and the weighted Bergman modules $L^2_{a,\alpha}(\mathbb{D})$ $(\alpha > -1, \alpha \in \mathbb{Z})$. It is known that a quotient module of an analytic Hilbert module is irreducible, that is, C_z does not have any non-trivial reducing subspace (cf. Theorem 3.3 and Lemma 3.4 in [4]). Using this, we obtain the next lemma.

Lemma 2.4. *Let $\mathcal{Q}$ be a non-zero quotient module of an analytic Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$. Then $[C_z, C_z^*] = 0$ if and only if $\mathcal{Q}$ is one dimensional.*

Proof. First note that for any non-zero quotient module $\mathcal{Q}$ of $\mathcal{H}$, the C^*-algebra $C^*(\mathcal{Q})$ is irreducible. If C_z is normal, then $C^*(\mathcal{Q}) \subseteq C^*(\mathcal{Q})' = \mathbb{C}I$. Thus $C^*(\mathcal{Q}) = \mathbb{C}I$, and therefore, $\mathcal{Q}$ is one dimensional. The converse part is trivial, and the proof follows. ☐

Let $\{k_i\}_{i=1}^n$ be positive definite functions on $\mathbb{D}$. Then $\mathcal{H}_K := \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}$ is said to be an *analytic Hilbert module over* $\mathbb{C}[\boldsymbol{z}]$ if $\mathcal{H}_{k_i}$ is an analytic Hilbert module over $\mathbb{C}[z]$ for all $i = 1, \ldots, n$. In this case, $\mathcal{H}_K \subseteq \mathcal{O}(\mathbb{D}^n, \mathbb{C})$ and

$$K(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n k_i(z_i, w_i) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

is the reproducing kernel function of $\mathcal{H}_K$ (cf. [4]). In the sequel, we will often identify M_{z_i} on $\mathcal{H}_K$ with the operator $I_{\mathcal{H}_{k_1}} \otimes \cdots \otimes \underbrace{M_z}_{\text{i-th place}} \otimes \cdots \otimes I_{\mathcal{H}_{k_n}}$, $i = 1, \ldots, n$, on the n-fold Hilbert space tensor product $\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}$. We end this section with a result on essential normality of a Beurling type quotient module $\mathcal{Q}_\theta$, where θ is a one variable inner function in $\mathbb{D}^n$.

Lemma 2.5. *Let $\theta \in H^\infty(\mathbb{D}^n)$ be a one variable inner function and $n \geq 3$. Then $\mathcal{Q}_\theta$ is not essentially normal.*

Proof. Without loss of generality we may assume that $\theta(\boldsymbol{z}) = \theta'(z_1)$ for some inner function $\theta' \in H^\infty(\mathbb{D})$. Then it follows that $\mathcal{S}_\theta = \mathcal{S}_{\theta'} \otimes H^2(\mathbb{D}^{n-1})$ and

$$\mathcal{Q}_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n) = \mathcal{Q}_{\theta'} \otimes H^2(\mathbb{D}^{n-1}).$$

Now we compute the self commutator of C_{z_2}:

$$\begin{aligned} [C_{z_2}, C_{z_2}^*] &= P_{\mathcal{Q}_\theta} M_{z_2} M_{z_2}^*|_{\mathcal{Q}_\theta} - P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{Q}_\theta} M_{z_2}|_{\mathcal{Q}_\theta} \\ &= P_{\mathcal{Q}_\theta} M_{z_2} M_{z_2}^*|_{\mathcal{Q}_\theta} - I_{\mathcal{Q}_\theta} + P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta} M_{z_2}|_{\mathcal{Q}_\theta}. \end{aligned}$$

Using the fact

$$P_{\mathcal{S}_\theta} M_{z_2}|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = (P_{\mathcal{S}_{\theta'}} \otimes I_{H^2(\mathbb{D})} \otimes I_{H^2(\mathbb{D}^{n-2})}) M_{z_2}|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = 0,$$

and

$$M_{z_2}^*|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = 0,$$

we conclude that

$$[C_{z_2}, C_{z_2}^*]|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = -I_{\mathcal{Q}_\theta}|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})} = -I_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})}.$$

Since $n \geq 3$, $[C_{z_2}, C_{z_2}^*]|_{\mathcal{Q}_{\theta'} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})}$ is not compact, and hence the commutator $[C_{z_2}, C_{z_2}^*]$ is not compact. This completes the proof. $\square$

3. Essential normality

Our purpose in this section is to prove a list of results concerning essential normality for certain classes of quotient modules. We begin with the class of Beurling type quotient modules of $H^2(\mathbb{D}^n)$, $n \geq 3$.

Theorem 3.1. *Let θ be an inner function in $H^\infty(\mathbb{D}^n)$ and $n \geq 3$. Then $\mathcal{Q}_\theta$ is not essentially normal.*

Proof. By Lemma 2.5, we may assume without loss of generality that θ depends on both z_1 and z_2 variables. We now show that $[C_{z_1}, C_{z_2}^*]$ is not compact. To see this, we compute

$$\begin{aligned}[C_{z_1}, C_{z_2}^*] &= P_{\mathcal{Q}_\theta} M_{z_1} M_{z_2}^*|_{\mathcal{Q}_\theta} - P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{Q}_\theta} M_{z_1}|_{\mathcal{Q}_\theta} = P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta} M_{z_1}|_{\mathcal{Q}_\theta} \\ &= P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta)} M_{z_1}|_{\mathcal{Q}_\theta} + P_{\mathcal{Q}_\theta} M_{z_2}^* P_{z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta} M_{z_1}|_{\mathcal{Q}_\theta}.\end{aligned}$$

Since M_{z_1} and M_{z_2} are isometries, we have

$$P_{\mathcal{Q}_\theta} M_{z_i}^* P_{z_i \mathcal{S}_\theta} = 0 \qquad (i = 1, 2).$$

This implies

$$P_{\mathcal{Q}_\theta} M_{z_2}^* P_{z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta} M_{z_1}|_{\mathcal{Q}_\theta} = 0,$$

and

$$[C_{z_1}, C_{z_2}^*] = P_{\mathcal{Q}_\theta} M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta)} M_{z_1}|_{\mathcal{Q}_\theta}.$$

On the other hand, since $\mathcal{S}_\theta = \theta H^2(\mathbb{D}^n)$, we have

$$\begin{aligned}\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta) &= \theta H^2(\mathbb{D}^n) \ominus \theta(z_1 H^2(\mathbb{D}^n) + z_2 H^2(\mathbb{D}^n)) \\ &= \theta(\mathbb{C} \otimes \mathbb{C} \otimes H^2(\mathbb{D}^{n-2})),\end{aligned}$$

and therefore,

$$M_{z_2}^*(\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta)) \subseteq \mathcal{Q}_\theta.$$

Consequently,

$$[C_{z_1}, C_{z_2}^*] = M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1 \mathcal{S}_\theta + z_2 \mathcal{S}_\theta)} M_{z_1}|_{\mathcal{Q}_\theta}.$$

By Lemma 2.1, it is enough to show that $\langle [C_{z_1}, C_{z_2}^*] K_{\boldsymbol{w}}, K_{\boldsymbol{w}} \rangle$ does not converge to 0 as w_j approaches to $\partial \mathbb{D}$ for some fixed $3 \leq j \leq n$, and keeping all other co-ordinates of $\boldsymbol{w} = (w_1, \ldots w_{j-1}, w_j, w_{j+1}, \ldots, w_n) \in \mathbb{D}^n$ fixed. To this end, let $\boldsymbol{w} \in \mathbb{D}^n$. Since $\{\theta z_3^{m_3} \cdots z_n^{m_n} :$

$m_3, \ldots, m_n \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)$, we have

$$\begin{aligned}
P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)}(z_2 K_{\boldsymbol{w}}) &= \sum_{m_3,\ldots,m_n \in \mathbb{N}} \langle z_2 K_{\boldsymbol{w}}, \theta z_3^{m_3} \cdots z_n^{m_n} \rangle \theta z_3^{m_3} \cdots z_n^{m_n} \\
&= \sum_{m_3,\ldots,m_n \in \mathbb{N}} \langle K_{\boldsymbol{w}}, z_3^{m_3} \cdots z_n^{m_n} (M_{z_2}^* \theta) \rangle \theta z_3^{m_3} \cdots z_n^{m_n} \\
&= \frac{1}{\|\mathbb{S}(\cdot, \boldsymbol{w})\|} \theta \sum_{m_3,\ldots,m_n \in \mathbb{N}} (\overline{w_3} z_3)^{m_3} \ldots (\overline{w_n} z_n)^{m_n} \overline{M_{z_2}^* \theta(\boldsymbol{w})} \\
&= \overline{M_{z_2}^* \theta(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2)^{\frac{1}{2}} \Big(\prod_{i=3}^{n} K_{w_i} \Big) \theta.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle [C_{z_1}, C_{z_2}^*] K_{\boldsymbol{w}}, K_{\boldsymbol{w}} \rangle &= \langle M_{z_2}^* P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)} M_{z_1} P_{\mathcal{Q}_\theta} K_{\boldsymbol{w}}, K_{\boldsymbol{w}} \rangle \\
&= \langle M_{z_1} P_{\mathcal{Q}_\theta} K_{\boldsymbol{w}}, P_{\mathcal{S}_\theta \ominus (z_1\mathcal{S}_\theta + z_2\mathcal{S}_\theta)}(z_2 K_{\boldsymbol{w}}) \rangle \\
&= \overline{(M_{z_2}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2)^{\frac{1}{2}} \Big\langle M_{z_1} P_{\mathcal{Q}_\theta} K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} \theta \Big\rangle \\
&= \overline{(M_{z_2}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2)^{\frac{1}{2}} \Big\langle M_{z_1} (1 - \overline{\theta(\boldsymbol{w})} \theta) K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} \theta \Big\rangle,
\end{aligned}$$

where the last equality follows from (2.2). Since $M_{z_1}^*(\prod_{i=3}^{n} K_{w_i}) = 0$ and $M_\theta^* M_\theta = I_{H^2(\mathbb{D}^n)}$, we have

$$\Big\langle M_{z_1} \theta K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} \theta \Big\rangle = \Big\langle \theta M_{z_1} K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} \theta \Big\rangle = \Big\langle K_{\boldsymbol{w}}, M_{z_1}^* (\prod_{i=3}^{n} K_{w_i}) \Big\rangle = 0.$$

Therefore,

$$\begin{aligned}
\langle [C_{z_1}, C_{z_2}^*] K_{\boldsymbol{w}}, K_{\boldsymbol{w}} \rangle &= \overline{(M_{z_2}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2)^{\frac{1}{2}} \Big\langle M_{z_1} K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} \theta \Big\rangle \\
&= \overline{(M_{z_2}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2)^{\frac{1}{2}} \Big\langle K_{\boldsymbol{w}}, \prod_{i=3}^{n} K_{w_i} (M_{z_1}^* \theta) \Big\rangle \\
&= \overline{(M_{z_2}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2)^{\frac{1}{2}} \ \Big(\overline{M_{z_1}^* \theta(\boldsymbol{w})} \frac{1}{\|\mathbb{S}(\cdot, \boldsymbol{w})\|} \prod_{i=3}^{n} \frac{1}{(1 - |w_j|^2)^{\frac{1}{2}}} \Big) \\
&= \overline{(M_{z_2}^* \theta)(\boldsymbol{w})} \ \overline{(M_{z_1}^* \theta)(\boldsymbol{w})} \prod_{j=1}^{2} (1 - |w_j|^2).
\end{aligned}$$

Since θ depends on both z_1 and z_2 variables, $M_{z_1}^*\theta$ and $M_{z_2}^*\theta$ are non-zero functions. Therefore it follows that there exists an $l \in \{3, \ldots, n\}$ such that the limit of

$$\overline{(M_{z_2}^*\theta)(\boldsymbol{w})}\ \overline{(M_{z_1}^*\theta)(\boldsymbol{w})} \prod_{j=1}^{2}(1 - |w_j|^2)$$

as w_l approaches to $\partial\mathbb{D}$ and keeping all other coordinates of $\boldsymbol{w}$ fixed, is a non-zero number. This completes the proof. $\square$

We now proceed to the case of doubly commuting quotient modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Let $\mathcal{Q}$ be a quotient module of an analytic Hilbert module $\mathcal{H}_K$ over $\mathbb{C}[\boldsymbol{z}]$. It is known that $\mathcal{Q}$ is doubly commuting (that is, $[C_{z_i}, C_{z_j}^*] = 0$ for all $1 \leq i < j \leq n$) if and only if $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ for some quotient module $\mathcal{Q}_i$ of $\mathcal{H}_{k_i}$, $i = 1, \ldots, n$ (see [4], [15] and [17]).

THEOREM 3.2. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of an analytic Hilbert module $\mathcal{H}_K = \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}$ over $\mathbb{C}[\boldsymbol{z}]$, $n \geq 2$. Then $\mathcal{Q}$ is essentially normal if and only if one of the following holds:*

(i) *$\mathcal{Q}$ is finite dimensional.*

(ii) *There exits an $i \in \{1, \ldots, n\}$ such that $\mathcal{Q}_i$ is an infinite dimensional essentially normal quotient module of $\mathcal{H}_{k_i}$, and $\mathcal{Q}_j \cong \mathbb{C}$ for all $j \neq i$.*

Proof. Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be an infinite dimensional essentially normal quotient module. Then at least one of $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ is infinite dimensional. Without loss of generality we assume that $\mathcal{Q}_n$ is infinite dimensional. For each $i = 1, \ldots, n$, we now compute the self-commutator:

$$\begin{aligned} [C_{z_i}, C_{z_i}^*] &= P_{\mathcal{Q}} M_{z_i} M_{z_i}^*|_{\mathcal{Q}} - P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} \\ &= P_{\mathcal{Q}_1} \otimes \cdots \otimes P_{\mathcal{Q}_{i-1}} \otimes \underbrace{[C_z, C_z^*]_i}_{\text{i-th place}} \otimes P_{\mathcal{Q}_{i+1}} \otimes \cdots \otimes P_{\mathcal{Q}_n}, \end{aligned} \tag{3.3}$$

where $[C_z, C_z^*]_i$ is the self-commutator corresponding to the quotient module $\mathcal{Q}_i$. Since $\mathcal{Q}_n$ is infinite dimensional, the compactness of $[C_{z_i}, C_{z_i}^*]$ implies that $[C_z, C_z^*]_i = 0$ for all $i = 1, \ldots, n-1$. Therefore, by Lemma 2.4, it follows that $\mathcal{Q}_i \cong \mathbb{C}$, $i = 1, \ldots, n-1$.
Finally, for $i = n$, the compactness of $[C_{z_n}, C_{z_n}^*] = P_{\mathcal{Q}_1} \otimes \cdots \otimes P_{\mathcal{Q}_{n-1}} \otimes [C_z, C_z^*]_n$ implies that $[C_z, C_z^*]_n$ is compact, that is, $\mathcal{Q}_n$ is essentially normal.

For the converse, it is enough to show that (ii) implies $\mathcal{Q}$ is essentially normal. Again, without loss of generality, we assume that $\mathcal{Q}_n$ is infinite dimensional essentially normal quotient module. Then it readily follows from (3.3) that $[C_{z_i}, C_{z_i}^*] = 0$, $i = 1, \ldots, n-1$, and $[C_{z_n}, C_{z_n}^*]$ is compact. Now the proof follows from Fuglede-Putnam theorem. $\square$

The above result applies, in particular, if $\mathcal{H}_K$ is the $H^2(\mathbb{D}^n)$ or the weighted Bergman modules $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$ ($\boldsymbol{\alpha} \in \mathbb{Z}^n, \alpha_i > -1, i = 1, \ldots, n$). Moreover, since every quotient module of $H^2(\mathbb{D})$ is essentially normal, by Theorem 3.2 we have the following corollary.

COROLLARY 3.3. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, $n \geq 2$. Then $\mathcal{Q}$ is essentially normal if and only if one of the following holds:*

(i) $\mathcal{Q}$ *is finite dimensional.*

(ii) *There exits an* $i \in \{1, \dots, n\}$ *such that* $\mathcal{Q}_i$ *is infinite dimensional, and* $\mathcal{Q}_j \cong \mathbb{C}$ *for all* $j \neq i$.

It is also well known that a quotient module $\mathcal{Q}$ of the Bergman module $L^2_a(\mathbb{D})$ is essentially normal if and only if

$$\dim(\mathcal{S} \ominus z\mathcal{S}) < \infty,$$

where $\mathcal{S} := L^2_a(\mathbb{D}) \ominus \mathcal{Q}$ is the corresponding submodule (see [20, Theorem 3.1]). Using this and Theorem 3.2, we have the following result.

Corollary 3.4. *Let* $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ *be a doubly commuting quotient module of* $L^2_a(\mathbb{D}^n)$, $n \geq 2$. *Then* $\mathcal{Q}$ *is essentially normal if and only if one of the following holds:*

(i) $\mathcal{Q}$ *is finite dimensional.*

(ii) *There exists an* $i \in \{1, \dots, n\}$ *such that* $\mathcal{Q}_i$ *is infinite dimensional with* $\dim(\mathcal{S}_i \ominus z\mathcal{S}_i) < \infty$ *and* $\mathcal{Q}_j \cong \mathbb{C}$ *for all* $j \neq i$, *where* $\mathcal{S}_i = L^2_a(\mathbb{D}) \ominus \mathcal{Q}_i$.

We now restrict our attention to $H^2(\mathbb{D}^2)$, and formulate the definition of the Rudin quotient module of $H^2(\mathbb{D}^2)$ (see [5], [8]). Let $\Psi = \{\psi_n\}_{n=0}^{\infty} \subseteq H^2(\mathbb{D})$ be an increasing sequence of finite Blaschke products and $\Phi = \{\varphi_n\}_{n=0}^{\infty} \subseteq H^2(\mathbb{D})$ be a decreasing sequence of Blaschke products, that is, ψ_{n+1}/ψ_n and φ_n/φ_{n+1} are non-constant inner functions for all $n \in \mathbb{N}$. Then the *Rudin quotient module* corresponding to Ψ and Φ is denoted by $\mathcal{Q}_{\Psi,\Phi}$, and defined by

$$\mathcal{Q}_{\Psi,\Phi} := \bigvee_{n=0}^{\infty} \left(\mathcal{Q}_{\psi_n} \otimes \mathcal{Q}_{\varphi_n}\right).$$

We denote by $\mathcal{S}_{\Psi,\Phi}$ the submodule $H^2(\mathbb{D}^2) \ominus \mathcal{Q}_{\Psi,\Phi}$ corresponding to $\mathcal{Q}_{\Psi,\Phi}$. The following representations of $\mathcal{Q}_{\Psi,\Phi}$ and $\mathcal{S}_{\Psi,\Phi}$ are very useful (see [5]):

$$\mathcal{Q}_{\Psi,\Phi} = \bigoplus_{n\geq 0} (\mathcal{Q}_{\psi_n} \ominus \mathcal{Q}_{\psi_{n-1}}) \otimes \mathcal{Q}_{\varphi_n} \quad \text{and} \quad \mathcal{S}_{\Psi,\Phi} = \mathcal{Q}' \otimes H^2(\mathbb{D}) \bigoplus_{n\geq 0} (\mathcal{Q}_{\psi_n} \ominus \mathcal{Q}_{\psi_{n-1}}) \otimes \mathcal{S}_{\varphi_n}, \tag{3.4}$$

where $\mathcal{Q}_{\psi_{-1}} := \{0\}$ and $\mathcal{Q}' = H^2(\mathbb{D}) \ominus \vee_{n\geq 0} \mathcal{Q}_{\psi_n}$.

Next we show that the Rudin quotient modules are not essentially normal.

Theorem 3.5. *Let* $\mathcal{Q}_{\Psi,\Phi}$ *be a Rudin quotient module of* $H^2(\mathbb{D}^2)$ *corresponding to an increasing sequence of finite Blaschke products* $\Psi = \{\psi_n\}_{n\geq 0}$ *and a decreasing sequence of Blaschke products* $\Phi = \{\varphi_n\}_{n\geq 0}$. *Then* $\mathcal{Q}_{\Psi,\Phi}$ *is not essentially normal.*

Proof. Let b_β, the Blaschke factor corresponding to $\beta \in \mathbb{D}$, be a factor of ψ_{m+1}/ψ_m for some $m \geq 0$. For contradiction, we assume that $\mathcal{Q}_{\Psi,\Phi}$ is essentially normal. Then, as ψ_m is a finite Blaschke product, $[C_{\psi_m(z_1)}, C^*_{\psi_m(z_1)}]$ is compact, where $C_{\psi_m(z_1)} = P_{\mathcal{Q}} M_{\psi_m(z_1)}|_{\mathcal{Q}}$ and $\mathcal{Q} := \mathcal{Q}_{\Psi,\Phi}$. Now setting $\mathcal{S} := \mathcal{S}_{\Psi,\Phi}$, we have

$$\begin{aligned}
[C_{\psi_m(z_1)}, C^*_{\psi_m(z_1)}] &= P_{\mathcal{Q}} M_{\psi_m(z_1)} M^*_{\psi_m(z_1)}|_{\mathcal{Q}} - P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{Q}} M_{\psi_m(z_1)}|_{\mathcal{Q}} \\
&= -P_{\mathcal{Q}}(I - M_{\psi_m(z_1)} M^*_{\psi_m(z_1)})|_{\mathcal{Q}} + P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{S}} M_{\psi_m(z_1)}|_{\mathcal{Q}} \\
&= -P_{\mathcal{Q}}(P_{\mathcal{Q}_{\psi_m}} \otimes I)|_{\mathcal{Q}} + P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{S}} M_{\psi_m(z_1)}|_{\mathcal{Q}}.
\end{aligned} \tag{3.5}$$

Since φ_{m+1} is an infinite Blaschke product, there exists a sequence $\{\lambda_i\} \subset \mathbb{D}$ such that $K_{\lambda_i} \in \mathcal{Q}_{\varphi_{m+1}}$ and λ_i approaches to $\partial\mathbb{D}$ as $i \to \infty$. Furthermore, since $K_\beta \otimes K_{\lambda_i} \in \mathcal{Q}$ and $\psi_m K_\beta \otimes K_{\lambda_i} \in (\mathcal{Q}_{\psi_{m+1}} \ominus \mathcal{Q}_{\psi_m}) \otimes \mathcal{Q}_{\varphi_{m+1}}$, we have $P_{\mathcal{S}}(\psi_m K_\beta \otimes K_{\lambda_i}) = 0$, $i = 1, \ldots, n$ (by (3.4)). Thus

$$P_{\mathcal{Q}} M^*_{\psi_m(z_1)} P_{\mathcal{S}} M_{\psi_m(z_1)}(K_\beta \otimes K_{\lambda_i}) = 0.$$

Finally, from (3.5), we have

$$\begin{aligned} \langle [C_{\psi_m(z_1)}, C^*_{\psi_m(z_1)}](K_\beta \otimes K_{\lambda_i}), K_\beta \otimes K_{\lambda_i}\rangle &= -\langle (P_{\mathcal{Q}_{\psi_m}} K_\beta) \otimes K_{\lambda_i}, K_\beta \otimes K_{\lambda_i}\rangle \\ &= -\langle (1 - \overline{\psi_m(\beta)}\psi_m) K_\beta, K_\beta\rangle \\ &= -(1 - |\psi_m(\beta)|^2), \end{aligned}$$

which does not converges to 0 as λ_i approaches to $\partial\mathbb{D}$. This completes the proof. $\square$

REMARK 3.6. *Let $m > 1$. For a decreasing Blaschke products $\{\varphi_n\}_{n=1}^m$ and an increasing finite Blaschke products $\{\psi_n\}_{n=1}^m$, we consider the quotient module*

$$\mathcal{Q} = \bigvee_{n=1}^{m} \mathcal{Q}_{\psi_n} \otimes \mathcal{Q}_{\varphi_n}.$$

Adapting the techniques in the proof of the above theorem, one can conclude that $\mathcal{Q}$ is essentially normal if and only if φ_n is finite Blaschke products for all $n = 1, \ldots, m$. In other words, $\mathcal{Q}$ is essentially normal if and only if $\mathcal{Q}$ is finite dimensional.

4. BOUNDARY REPRESENTATIONS FOR DOUBLY COMMUTING QUOTIENT MODULES

In this section we study boundary representations for doubly commuting quotient modules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. First, we prove a general result in the setting of minimal tensor products of C^*-algebras. Before that we fix some notations. We denote by $V_1 \underline{\otimes} V_2$ the algebraic tensor product of two vector spaces V_1 and V_2, and by $A_1 \otimes A_2$ the minimal tensor product of two C^*-algebras A_1 and A_2.

THEOREM 4.1. *Let A_i be a unital subalgebra of $\mathcal{B}(\mathcal{H}_i)$ for some Hilbert space $\mathcal{H}_i$, and let $C^*(A_i)$ be the irreducible C^*-algebra generated by A_i in $\mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$. Set $A := \overline{(A_1 \underline{\otimes} A_2)}$, the norm closure of $A_1 \underline{\otimes} A_2$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then the following are equivalent.*

(i) *The identity representation of $C^*(A_1) \otimes C^*(A_2)$ is a boundary representation relative to A.*
(ii) *The identity representation of $C^*(A_i)$ is a boundary representation relative to A_i for all $i = 1, 2$.*

Proof. Suppose (i) holds, and for contradiction, we assume that the identity representation of $C^*(A_1)$ is not a boundary representation relative to A_1. Then there exists a CP map $\tau : C^*(A_1) \to \mathcal{B}(\mathcal{H}_1)$ different from $\mathrm{id}_{C^*(A_1)}$, but $\tau = \mathrm{id}_{C^*(A_1)}$ on A_1. Then the CP map

$$\tau \otimes \mathrm{id}_{C^*(A_2)} : C^*(A_1) \otimes C^*(A_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2),$$

is a CP extension of the map $\mathrm{id}_{C^*(A_1)\otimes C^*(A_2)}|_A$ to $C^*(A_1) \otimes C^*(A_2)$, and $\tau \otimes \mathrm{id}_{C^*(A_2)} \neq \mathrm{id}_{C^*(A_1)\otimes C^*(A_2)}$. This is a contradiction.

On the other hand, suppose that (ii) holds. It follows from [1, Theorem 2.2.7] that the identity representation of $C^*(A_1)\otimes C^*(A_2)$ is a boundary representation relative to the linear subspace

$$\mathbb{C}I \otimes C^*(A_1) \bigvee C^*(A_1) \otimes \mathbb{C}I.$$

Thus it is enough to show that any CP extension $\tau : C^*(A_1) \otimes C^*(A_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of $\mathrm{id}_{C^*(A_1)\otimes C^*(A_2)}|_A$ agrees with $\mathrm{id}_{C^*(A_1)\otimes C^*(A_2)}$ on the subspace $\mathbb{C}I \otimes C^*(A_1) \bigvee C^*(A_1) \otimes \mathbb{C}I$, that is,

$$\tau(I \otimes T_2) = I \otimes T_2 \ \text{ and } \tau(T_1 \otimes I) = T_1 \otimes I \qquad (T_1 \in C^*(A_1), T_2 \in C^*(A_2)).$$

To this end, for $h \in \mathcal{H}_1$, let $\omega_{h,h}$ be the positive linear functional on $\mathcal{B}(\mathcal{H}_1)$ defined by $T \mapsto \langle Th, h\rangle$, and

$$(\omega_{h,h}\overline{\otimes}\, \mathrm{id}_{\mathcal{B}(\mathcal{H}_2)}) \circ \tau : C^*(A_1) \otimes C^*(A_2) \to \mathcal{B}(\mathcal{H}_2)$$

be the corresponding CP map. By identifying $\mathbb{C}I \otimes C^*(A_2)$ with $C^*(A_2)$, one sees that the CP map $(\omega_{h,h}\overline{\otimes}\, \mathrm{id}_{\mathcal{B}(\mathcal{H}_2)}) \circ \tau|_{\mathbb{C}I\otimes C^*(A_2)}$ is an extension of $\omega_{h,h}(I)\, \mathrm{id}_{C^*(A_2)}|_{A_2}$. Then, by the assumption, we have

$$(\omega_{h,h}\overline{\otimes}\, \mathrm{id}_{\mathcal{B}(\mathcal{H}_2)}) \circ \tau(I \otimes T) = \langle h, h\rangle T = \omega_{h,h}\overline{\otimes}\, \mathrm{id}_{C^*(A_2)}(I \otimes T) \quad (T \in C^*(A_2)).$$

By linearity, the above equality is also true for any linear functional $\omega_{h,k}$ on $\mathcal{B}(\mathcal{H}_1)$ for $h, k \in \mathcal{H}_1$. Therefore

$$\tau = \mathrm{id}_{C^*(A_1)\otimes C^*(A_2)} \ \text{ on } \mathbb{C}I \otimes C^*(A_2).$$

Similarly, by considering linear functionals on $\mathcal{B}(\mathcal{H}_2)$ and repeating the above arguments, we have that

$$\tau = \mathrm{id}_{C^*(A_1)\otimes C^*(A_2)} \ \text{ on } C^*(A_1) \otimes \mathbb{C}I.$$

This completes the proof. ∎

Remark 4.2. *The above result can be easily generalized for finite number of irreducible generating C^*-algebras corresponding to unital subalgebras.*

As a straightforward consequence of Remark 4.2 we obtain the following:

Theorem 4.3. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}$ over $\mathbb{C}[\boldsymbol{z}]$. Then the following are equivalent.*

(i) *The identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$.*
(ii) *The identity representation of $C^*(\mathcal{Q}_i)$ is a boundary representation relative to $B(\mathcal{Q}_i)$ for all $i = 1, \ldots, n$.*

Proof. The result follows from Remark 4.2 and the fact that

$$C^*(\mathcal{Q}) = C^*(\mathcal{Q}_1) \otimes \cdots \otimes C^*(\mathcal{Q}_n),$$

and

$$B(\mathcal{Q}) = \overline{B(\mathcal{Q}_1)\underline{\otimes} \cdots \underline{\otimes} B(\mathcal{Q}_n)},$$

where the closure is in the norm topology of $B(\mathcal{Q})$. ∎

The following result is now an immediate consequence of Theorems 1.2 and 4.3.

Theorem 4.4. *Let $\mathcal{Q} = \mathcal{Q}_{\theta_1} \otimes \cdots \otimes \mathcal{Q}_{\theta_n}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, where θ_i, $i = 1, \ldots, n$, is a one variable inner function. Then the following are equivalent.*

(i) *The identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$.*
(ii) *The identity representation of $C^*(\mathcal{Q}_{\theta_i})$ is a boundary representation relative to $B(\mathcal{Q}_{\theta_i})$ for all $i = 1, \ldots, n$.*
(iii) *For all $i = 1, \ldots, n$, Z_{θ_i} is a proper subset of $\mathbb{T}$, where Z_{θ_i} consists of all points λ on $\mathbb{T}$ for which θ_i cannot be continued analytically from $\mathbb{D}$ to λ.*

Now we turn to the case of the Bergman module $L^2_a(\mathbb{D}^n)$. For $n = 1$, boundary representations corresponding to a quotient module of $L^2_a(\mathbb{D})$ are studied in [14]. For a submodule $\mathcal{S}$ of $L^2_a(\mathbb{D})$, set

$$Z_*(\mathcal{S}) := \bigcap_{f \in \mathcal{S}} Z_*(f),$$

where

$$Z_*(f) = \big\{\lambda \in \mathbb{D} : f(\lambda) = 0\big\} \cup \big\{\lambda \in \mathbb{T} : \liminf_{z \in \mathbb{D}, z \to \lambda} |f(z)| = 0\big\}.$$

It is easy to see that for a finite dimensional quotient module $\mathcal{Q}$ of $L^2_a(\mathbb{D})$, the identity representation of $C^*(\mathcal{Q})$ is always a boundary representation relative to $B(\mathcal{Q})$. On the other hand, for an infinite dimensional $\mathcal{Q}$, the identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$ if and only if $\dim(\mathcal{S} \ominus z\mathcal{S}) = 1$ and $Z_*(\mathcal{S})$ is a proper subset of $\mathbb{T}$, where $\mathcal{S} = L^2_a(\mathbb{D}) \ominus \mathcal{Q}$ is the corresponding submodule (see [14, Theorem 1.2]). Using this and Theorem 4.3, we have the following result.

Theorem 4.5. *Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of $L^2_a(\mathbb{D}^n)$. Then the following are equivalent.*

(i) *The identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $B(\mathcal{Q})$.*
(ii) *The identity representation of $C^*(\mathcal{Q}_i)$ is a boundary representation relative to $B(\mathcal{Q}_i)$ for all $i = 1, \ldots, n$.*
(iii) *If $\mathcal{Q}_i$ $(1 \leq i \leq n)$ is infinite dimensional then dim$(\mathcal{S}_i \ominus z\mathcal{S}_i) = 1$ and $Z_*(\mathcal{S}_i)$ is a proper subset of $\mathbb{T}$, where $\mathcal{S}_i = L^2_a(\mathbb{D}) \ominus \mathcal{Q}_i$ is the corresponding submodule.*

5. Boundary representations for $\mathcal{Q}_{\boldsymbol{\eta}}$-type quotient modules

In this section we study boundary representations for a special class of essentially normal quotient modules of $H^2(\mathbb{D}^n)$. If $\mathcal{Q}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$ we denote by $\sigma_e(\mathcal{Q})$ the essential joint spectrum of $(C_{z_1}, \ldots, C_{z_n})$. The following lemma is a standard application of Arveson's theory on boundary representations [1, 2].

Lemma 5.1. *Let $\mathcal{Q}$ be an essentially normal quotient module of $H^2(\mathbb{D}^n)$.*

(a) *If there exists a matrix-valued polynomial p such that*

$$\|p(C_{z_1}, \ldots, C_{z_n})\| > \|p\|^{\infty}_{\sigma_e(\mathcal{Q})},$$

then the identity representation of $C^*(\mathcal{Q})$ is a boundary representation relative to $\mathcal{B}(\mathcal{Q})$.

(b) *If the commuting tuple $(C_{z_1}, \dots, C_{z_n})$ has a normal dilation on $\sigma_e(\mathcal{Q})$, then the identity representation of $C^*(\mathcal{Q})$ is not a boundary representation relative to $\mathcal{B}(\mathcal{Q})$.*

Proof. a) Since $\mathcal{Q}$ is essentially normal, we have the following extension

$$0 \longrightarrow K(\mathcal{Q}) \hookrightarrow C^*(\mathcal{Q}) \longrightarrow C(\sigma_e(\mathcal{Q})) \longrightarrow 0.$$

If there exists a matrix-valued polynomial p such that

$$\|p(C_{z_1}, \dots, C_{z_n})\| > \|p\|^\infty_{\sigma_e(\mathcal{Q})},$$

then the restriction of the canonical contractive homomorphism

$$q : C^*(\mathcal{Q}) \to C^*(\mathcal{Q})/K(\mathcal{Q}) \cong C(\sigma_e(\mathcal{Q}))$$

to $\mathcal{B}(\mathcal{Q})$ is not a complete isometry. The desired conclusion now follows from [2, Theorem 2.1.1].

(b) The existence of a normal dilation implies that the above completely contractive map q restricted to the linear span of $\mathcal{B}(\mathcal{Q}) \cup \mathcal{B}(\mathcal{Q})^*$ is a complete isometry. Then the conclusion again follows from [2, Theorem 2.1.1]. ☐

We now define a $\mathcal{Q}_{\boldsymbol{\eta}}$-type quotient module of $H^2(\mathbb{D}^n)$. Let $\{\eta_j\}_{j=1}^n \subseteq H^\infty(\mathbb{D})$ be one variable inner functions. Notice that $\boldsymbol{\eta}_j$, defined by

$$\boldsymbol{\eta}_j(\boldsymbol{z}) := \eta_j(z_j) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

is a one variable (in z_j) inner function on $\mathbb{D}^n$, $j = 1, \dots, n$. Let $\mathcal{S}_{\boldsymbol{\eta}}$ be the submodule of $H^2(\mathbb{D}^n)$ generated by $\{\boldsymbol{\eta}_{j-1} - \boldsymbol{\eta}_j\}_{j=2}^n$, that is,

$$\mathcal{S}_{\boldsymbol{\eta}} = \overline{\text{span}}\{\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{\eta}_2 - \boldsymbol{\eta}_3, \dots, \boldsymbol{\eta}_{n-1} - \boldsymbol{\eta}_n\}.$$

Let $\mathcal{Q}_{\boldsymbol{\eta}}$ be the corresponding quotient module of $\mathcal{S}_{\boldsymbol{\eta}}$, that is,

$$\mathcal{Q}_{\boldsymbol{\eta}} = \mathcal{S}_{\boldsymbol{\eta}}^\perp \cong H^2(\mathbb{D}^n)/\mathcal{S}_{\boldsymbol{\eta}}.$$

Set

$$V_{\boldsymbol{\eta}} = \{\boldsymbol{z} \in \mathbb{D}^n : \eta_1(z_1) = \cdots = \eta_n(z_n)\},$$

and

$$\partial V_{\boldsymbol{\eta}} = \{\boldsymbol{z} \in \partial\mathbb{D}^n : \eta_1(z_1) = \cdots = \eta_n(z_n)\}.$$

The zero sets associated to $\mathcal{S}_{\boldsymbol{\eta}}$ are defined by

$$Z(\mathcal{S}_{\boldsymbol{\eta}}) = \{\boldsymbol{z} \in \mathbb{D}^n : f(\boldsymbol{z}) = 0, \text{ for all } f \in \mathcal{S}_{\boldsymbol{\eta}}\},$$

and

$$Z_\partial(\mathcal{S}_{\boldsymbol{\eta}}) = \left\{\boldsymbol{z} \in \partial\mathbb{D}^n : \lim_{k\to\infty} \boldsymbol{z}_k = \boldsymbol{z} \text{ for some } \{\boldsymbol{z}_k\} \subset Z(\mathcal{S}_{\boldsymbol{\eta}})\right\}.$$

Then it easily follows from the definition that

$$Z(\mathcal{S}_{\boldsymbol{\eta}}) = V_{\boldsymbol{\eta}}, \qquad \text{and} \qquad Z_\partial(\mathcal{S}_{\boldsymbol{\eta}}) = \partial V_{\boldsymbol{\eta}}.$$

The following characterization of essentially normal $\mathcal{Q}_{\boldsymbol{\eta}}$-type quotient modules is due to Clark [7] and Wang [19].

THEOREM 5.2. *A quotient module $\mathcal{Q}_{\boldsymbol{\eta}}$ is essentially normal if and only if η_i is a finite Blaschke product for all $i = 1, \dots, n$.*

We now present a lemma that relates the essential spectrum of $\mathcal{Q}_{\boldsymbol{\eta}}$ to the boundary of $V_{\boldsymbol{\eta}}$.

LEMMA 5.3. *If $\mathcal{Q}_{\boldsymbol{\eta}}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$, then*

$$\sigma_e(\mathcal{Q}_{\boldsymbol{\eta}}) = \partial V_{\boldsymbol{\eta}}.$$

Proof. By Theorem 5.2 we have that η_i, $i = 1, \ldots, n$, is a finite Blaschke product. Hence, together with [13, Theorem 6.1], we get

$$Z_\partial(\mathcal{S}_{\boldsymbol{\eta}}) = \partial V_{\boldsymbol{\eta}} \subseteq \sigma_e(\mathcal{Q}_{\boldsymbol{\eta}}).$$

On the other hand, by Theorems 5.1 and 7.1 in [7], $(C_{z_1}, \ldots, C_{z_n})$ on $\mathcal{Q}_{\boldsymbol{\eta}}$ is unitarily equivalent to the co-ordinate multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ on $A^{2,n-2}(V_{\boldsymbol{\eta}})$, and

$$\sigma_e(M_{z_i}) \subseteq \mathbb{T}.$$

Thus

$$\sigma_e(\mathcal{Q}_{\boldsymbol{\eta}}) \subseteq \mathbb{T}^n.$$

As $\boldsymbol{\eta}_j$ is a z_i-variable finite Blaschke product, for all $j = 1, \ldots, n$, it follows that

$$(\boldsymbol{\eta}_i(\boldsymbol{z}) - \boldsymbol{\eta}_{i+1}(\boldsymbol{z}))(C_{z_1}, \ldots, C_{z_n}) = 0,$$

for all $i = 1, \ldots, n-1$. By spectral mapping theorem, it follows that

$$\boldsymbol{\eta}_i(\boldsymbol{z}) = \boldsymbol{\eta}_{i+1}(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \sigma_e(\mathcal{Q}_{\boldsymbol{\eta}})),$$

for each $i = 1, \ldots, n-1$. Thus

$$\sigma_e(\mathcal{Q}_{\boldsymbol{\eta}}) \subseteq \mathbb{T}^n \cap V_{\boldsymbol{\eta}} = \partial V_{\boldsymbol{\eta}},$$

which completes the proof. □

We are now ready to prove the main result of this section concerning boundary representations of $\mathcal{Q}_{\boldsymbol{\eta}}$-type quotient modules.

THEOREM 5.4. *If $\mathcal{Q}_{\boldsymbol{\eta}}$ is an essentially normal quotient module of $H^2(\mathbb{D}^n)$, then the identity representation of $C^*(\mathcal{Q}_{\boldsymbol{\eta}})$ is not a boundary representation relative to $B(\mathcal{Q}_{\boldsymbol{\eta}})$.*

Proof. By virtue of Lemmas 5.1 and 5.3 it suffices to prove that the tuple $(C_{z_1}, \ldots, C_{z_n})$ has a normal dilation on $\partial V_{\boldsymbol{\eta}}$. Note again that, according to [7, Theorem 5.1], the tuple $(C_{z_1}, \ldots, C_{z_n})$ is unitary equivalent to the n-tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ on $A^{2,n-2}(V_{\boldsymbol{\eta}})$. Now on account of the appendix of this article, there exists a measure μ on $V_{\boldsymbol{\eta}}$ such that

$$A^{2,n-2}(V_{\boldsymbol{\eta}}) \subset L^2(d\mu).$$

Consequently $(M_{z_1}, \ldots, M_{z_n})$ has a normal dilation on $V_{\boldsymbol{\eta}}$. Since $\partial V_{\boldsymbol{\eta}}$ is the Shilov boundary of $V_{\boldsymbol{\eta}}$, $(C_{z_1}, \ldots, C_{z_n})$ has a normal dilation on $\partial V_{\boldsymbol{\eta}}$. This completes the proof. □

The above result will be used in the proof of the main result, Theorem 6.3, in Section 6, on boundary representations of homogeneous quotient modules of $H^2(\mathbb{D}^2)$.

6. Boundary representations for homogeneous quotient modules

The purpose of this section is to investigate boundary representations for homogeneous quotient modules of $H^2(\mathbb{D}^2)$. First we recall a characterization of homogeneous polynomial in $\mathbb{C}[z_1, z_2]$ (see [13]). Given a homogeneous polynomial $p \in \mathbb{C}[z_1, z_2]$ there exist homogeneous polynomials $p_1, p_2 \in \mathbb{C}[z_1, z_2]$, unique up to a scalar multiple of modulus one, such that

$$p = p_1 p_2,$$

and

$$Z(p_1) \cap \partial\mathbb{D}^2 \subset \mathbb{T}^2 \quad \text{and} \quad Z(p_2) \cap \partial\mathbb{D}^2 \subset (\mathbb{D} \times \mathbb{T}) \cup (\mathbb{T} \times \mathbb{D}).$$

Let $[p]$ denote the submodule of $H^2(\mathbb{D}^2)$ generated by p. Suppose that $\mathcal{Q}_p$ is the corresponding quotient module of $H^2(\mathbb{D}^2)$, that is,

$$\mathcal{Q}_p = H^2(\mathbb{D}^2)/[p].$$

The following characterization of essential normality of $\mathcal{Q}_p$ is due to Guo and Wang [13].

Theorem 6.1. *Let p be a non-zero homogeneous polynomial in $\mathbb{C}[z_1, z_2]$, and $p = p_1 p_2$ be the factorization of p as above. Then the quotient module $\mathcal{Q}_p$ is essentially normal if and only if p_2 has one of the following forms:*

(i) $p_2 \equiv c$ *with* $c \neq 0$,
(ii) $p_2 = \alpha z_1 + \beta z_2$ *with* $|\alpha| \neq |\beta|$,
(iii) $p_2 = c(z_1 - \alpha z_2)(z_2 - \beta z_1)$ *with* $|\alpha| < 1, |\beta| < 1$ *and* $c \neq 0$.

The following result in [13] gives a description of the essential joint spectrum of the above type of quotient modules. For a proof we refer to the reader to [13, Theorem 6.2].

Lemma 6.2 (Guo & Wang, [13]). *Let p be a homogeneous polynomial. Then $\sigma_e(\mathcal{Q}_p) = Z(p) \cap \partial\mathbb{D}^2 = Z(p) \cap \mathbb{T}^2$.*

For our present purposes, however, we need only the fact that $\sigma_e(\mathcal{Q}_p) \subset Z(p)$.

We now state our main result of this section. This gives a characterization for the identity representation to be a boundary representation.

Theorem 6.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be a homogeneous polynomial. Suppose that $\mathcal{Q}_p$ is an essentially normal quotient module of $H^2(\mathbb{D}^2)$. Then the identity representation of $C^*(\mathcal{Q}_p)$ is a boundary representation relative to $B(\mathcal{Q}_p)$ if and only if p is not of the following form:*
(i) $p = c(z_1^m - \alpha z_2^m)$ *for some* $m \in \mathbb{N}$, $c \neq 0$ *and* $|\alpha| = 1$,
(ii) $p = \alpha z_1 + \beta z_2$ *with* $|\alpha| \neq |\beta|$.

Our proof of Theorem 6.3 on boundary representations for homogeneous quotient modules is based on the following two special cases. A couple of lemmas below describes these.

Lemma 6.4. *Let $p = c\prod_{i=1}^{m}(z_1 - \alpha_i z_2)^{n_i}$ be a homogeneous polynomial with $c \neq 0$ and α_i's are distinct scalars of modulus one. Assume further that $n_i > 1$ for some $i = 1, \ldots, m$. Then the identity representation of $C^*(\mathcal{Q}_p)$ is a boundary representation relative to $B(\mathcal{Q}_p)$.*

Proof. Without loss of any generality assume that $n_1 > 1$. Set

$$q := (z_1 - \alpha_1 z_2) \prod_{i=2}^{m} (z_1 - \alpha_i z_2)^{n_i}.$$

Then $q(C_{z_1}, C_{z_2})$ is a non-zero operator and $\|q\|^{\infty}_{Z(p)} = 0$, and hence, according to part (a) of Lemma 5.1, the identity representation of $C^*(\mathcal{Q}_p)$ is a boundary representation relative to $B(\mathcal{Q}_p)$. ☐

LEMMA 6.5. *Let $p = c(z_1 - \alpha z_2)$, for some $\alpha \in \mathbb{C}$ with $|\alpha| \neq 1$. Then the identity representation of $C^*(\mathcal{Q}_p)$ is not a boundary representation relative to $B(\mathcal{Q}_p)$.*

Proof. Let $\alpha = 0$. Then $\mathcal{Q}_p$ is unitarily equivalent to $H^2(\mathbb{D})$ and hence the conclusion follows easily. Now let $\alpha \neq 0$. Then

$$\frac{1}{\alpha} C_{z_1} = C_{z_2},$$

and hence $C^*(\mathcal{Q}_p)$ is generated by C_{z_1}. Now assume that $|\alpha| > 1$ (the $|\alpha| < 1$ case is similar). Set $\beta = \frac{\alpha}{|\alpha|^2}$ and

$$c_n := (\sum_{m=0}^{n} |\beta|^{2m})^{1/2} \qquad (n \in \mathbb{N}).$$

It follows that the sequence of homogeneous polynomials $\{p_n\}$ is an orthonormal basis for $\mathcal{Q}_p$ [10], where

$$p_n = \frac{1}{c_n} \sum_{m=0}^{n} z_1^m (\beta z_2)^{n-m} \quad (n \in \mathbb{N})$$

Moreover (again see [10])

$$\begin{aligned} C_{z_1}(p_n) &= P_{\mathcal{Q}_p}(\frac{1}{c_n} \sum_{m=0}^{n} z_1^{m+1} (\beta z_2)^{n-m}) \\ &= \frac{1}{c_n} \langle p_{n+1}, \sum_{m=0}^{n} z_1^{m+1} (\beta z_2)^{n-m} \rangle \ p_{n+1} \\ &= \frac{c_n}{c_{n+1}} p_{n+1}, \end{aligned}$$

for all $n \geq 0$, that is, C_{z_1} is a weighted shift with weights $\{\frac{c_n}{c_{n+1}}\}_{n \geq 0}$. Finally, since $\limsup_n \frac{c_n}{c_{n+1}} = \sup_n \frac{c_n}{c_{n+1}}$, the result follows from [2, Corollary 2]. ☐

We are now in a position to give a proof of Theorem 6.3.

Proof of Theorem 6.3. If $p = c(z_1^m - \alpha z_2^m)$, then $\mathcal{Q}_p$ is of the form $\mathcal{Q}_{\boldsymbol{\eta}}$, where $\eta_1 = z^m$ and $\eta_2 = \alpha z^m$. Now it follows from Theorem 5.4 that identity representation of $C^*(\mathcal{Q}_p)$ is not a boundary representation relative to $B(\mathcal{Q}_p)$. Also if $p = \alpha z_1 + \beta z_2$, then the required conclusion follows from Lemma 6.5.

For the necessary part, we fist note that, since $\mathcal{Q}_p$ is essentially normal, p can be represented as one of those three representations as in Theorem 6.1. Now by Lemma 6.4, is it enough to consider the case $p = p_1 p_2$, where

$$p_1 = \prod_{i=1}^{m} (z_1 - \alpha_i z_2),$$

$\{\alpha_i\}_{i=1}^m$ are all distinct scalar of modulus one, and p_2 is as in Theorem 6.1. In view of the forms of p_2 in Theorem 6.1, we next consider the following four cases.

Case I: Let $p_2 = c$, $p_1 = \prod_{i=1}^m (z_1 - \alpha_i z_2)$, $\{\alpha_i\}_{i=1}^m$ are all distinct scalar of modulus one, and that $p = p_1 p_2$ is not of the form $c(z_1^m - \alpha z_2^m)$.
In this case we have $m > 1$. Set

$$q = \frac{1}{z_1} \Big(\prod_{i=1}^{m} (z_1 - \alpha_i z_2) - (-1)^m \big(\prod_{i=1}^{m} \alpha_i\big) z_2^m \Big).$$

Then

$$q = \sum_{k=0}^{m-1} (-1)^k \left(\sum_{1 \le i_1 < \cdots < i_k \le m} \alpha_{i_1} \ldots \alpha_{i_k} \right) z_1^{m-k-1} z_2^k. \tag{6.6}$$

A simple calculation shows that

$$\|q\|^{\infty}_{Z(p) \cap \partial \mathbb{D}^2} = 1.$$

On the other hand

$$\|q(C_{z_1}, C_{z_2})\| \ge \|q\|_{H^2(\mathbb{D}^2)} = \sqrt{1 + \sum_{k=1}^{m-1} \left| \sum_{1 \le i_1 < \cdots < i_k \le m} \alpha_{i_1} \ldots \alpha_{i_k} \right|^2}.$$

Now if

$$\sum_{k=1}^{m-1} \left| \sum_{1 \le i_1 < \cdots < i_k \le m} \alpha_{i_1} \ldots \alpha_{i_k} \right|^2 = 0,$$

then (6.6) yields that

$$p_1 = \prod_{i=1}^{m} (z_1 - \alpha_i z_2) = z_1^m - \alpha z_2^m,$$

for some α of modulus one which is an obvious contradiction. Thus $\|q(C_{z_1}, C_{z_2})\| > 1$, and therefore, by Lemma 6.2 and Lemma 5.1, the identity representation is a boundary representation in this case.

Case II: Let $p_2 = (z_1 - \gamma z_2)$, $|\gamma| \neq 1$, $p_1 = \prod_{i=1}^m (z_1 - \alpha_i z_2)$ and $\{\alpha_i\}_{i=1}^m$ are distinct scalars of modulus one. Also, without loss of generality, we may assume that $|\gamma| < 1$. Otherwise, by interchanging the role of z_1 and z_2, we can consider that $p_2 = (z_2 - \frac{1}{\gamma} z_1)$ and $p_1 = \prod_{i=1}^m (z_2 - \frac{1}{\alpha_i} z_1)$. As in the previous case, set

$$q = \frac{1}{z_1} \Big(\prod_{i=1}^{m+1} (z_1 - \alpha_i z_2) - (-1)^{m+1} \prod_{i=1}^{m+1} \alpha_i z_2^{m+1} \Big),$$

where $\alpha_{m+1}=\gamma$. A similar computation shows that

$$\|q\|^{\infty}_{Z(p)\cap\partial\mathbb{D}^2}=1,$$

and

$$\|q(C_{z_1},C_{z_2})\|\geq\sqrt{1+\sum_{k=1}^{m}\Bigg|\sum_{1\leq i_1<\cdots<i_k\leq m+1}\alpha_{i_1}\ldots\alpha_{i_k}\Bigg|^2}.$$

Therefore, if

$$\sum_{k=1}^{m}\Bigg|\sum_{1\leq i_1<\cdots<i_k\leq m+1}\alpha_{i_1}\ldots\alpha_{i_k}\Bigg|^2=0,$$

then

$$\prod_{i=1}^{m+1}(z_1-\alpha_i z_2)=z_1^{m+1}-\alpha z_2^{m+1},$$

for some scalar α with $|\alpha|\neq 1$. Consequently

$$|\alpha_1|=\cdots=|\alpha_{m+1}|=|\alpha|^{1/(m+1)}\neq 1,$$

which is a contradiction. Therefore $\|q(C_{z_1},C_{z_2})\|>1$, and the conclusion again follows from Lemma 6.2 and Lemma 5.1.

Case III: Let $p=p_2=(z_1-\gamma_1 z_2)(z_1-\gamma_2 z_2)$, $|\gamma_1|<1$ and $|\gamma_2|>1$. Consider for each $\epsilon>0$

$$q_\epsilon:=(z_1-\epsilon\gamma_2 z_2)\in\mathbb{C}[z_1,z_2].$$

Set

$$V_1=\{(\gamma_1 z_2,z_2)\in\partial\mathbb{D}^2:|z_2|=1\}\text{ and }V_2=\{(\gamma_2 z_2,z_2)\in\partial\mathbb{D}^2:|\gamma_2 z_2|=1\}.$$

Note that

$$Z(p)\cap\partial\mathbb{D}^2=V_1\cup V_2,$$

and

$$\|q_\epsilon\|^{\infty}_{V_1}=|\gamma_1-\epsilon\gamma_2|,\quad \|q_\epsilon\|^{\infty}_{V_2}=(1-\epsilon).$$

Therefore for a sufficiently small $0<\epsilon<1$ we have

$$\|q_\epsilon\|^{\infty}_{Z(p)\cap\partial\mathbb{D}^2}<1.$$

On the other hand, for any $0<\epsilon<1$

$$\|q_\epsilon(C_{z_1},C_{z_2})\|\geq\sqrt{1+|\epsilon\gamma_2|^2}>1,$$

and the conclusion again follows from Lemma 5.1.

Case IV: Let $p_2=(z_1-\gamma_1 z_2)(z_1-\gamma_2 z_2)$, $|\gamma_1|<1$, $|\gamma_2|>1$, $p_1=\prod_{i=1}^{m}(z_1-\alpha_i z_2)$, and $\{\alpha_i\}_{i=1}^m$ are distinct scalars of modulus one. Set

$$V_{\gamma_1}=\{(\gamma_1 z_2,z_2)\in\partial\mathbb{D}^2:|z_2|=1\},\quad V_{\gamma_2}=\{(\gamma_2 z_2,z_2)\in\partial\mathbb{D}^2:|\gamma_2 z_2|=1\},$$

and

$$V_{\alpha_i}:=\{(\alpha_i z_2,z_2)\in\partial\mathbb{D}^2:|z_2|=1\},\quad i=1,\ldots,m.$$

Consider for $\epsilon>0$

$$q_\epsilon=z_2(z_1^m-\epsilon q')\in\mathbb{C}[z_1,z_2],$$

where

$$q' = \sum_{k=1}^{m} \left(\sum_{1 \le i_1 < \cdots < i_k \le m} \alpha_{i_1} \dots \alpha_{i_k} \right) z_1^{m-k} z_2^k.$$

Note that

$$\|q_\epsilon\|^\infty_{V_{\gamma_1}} \le |\gamma_1^m| - \epsilon M, \quad \|q_\epsilon\|^\infty_{V_{\gamma_2}} \le |\frac{1}{\gamma_2}| - \epsilon M,$$

and

$$\|q_\epsilon\|^\infty_{V_{\alpha_i}} = |(1-\epsilon)|, \quad i = 1, \dots, m,$$

where

$$M = \max\{|q'(\gamma_1, 1)|, |q'(\frac{1}{\gamma_2}, 1)|\}.$$

We now choose $0 < \epsilon < 1$ so that $\|q_\epsilon\|^\infty_{Z(p)\cap\partial\mathbb{D}^2} < 1$. On the other hand, since $\|q_\epsilon(C_{z_1}, C_{z_2})\| \ge \|q_\epsilon\| > 1$, the conclusion follows immediately. $\square$

Appendix: A measure on $V_{\boldsymbol{\eta}}$

In this appendix we define $A^{2,n}(V_{\boldsymbol{\eta}})$ following D. Clark [7], and present an explicit Borel measure α on $V_{\boldsymbol{\eta}}$ such that $A^{2,n}(V_{\boldsymbol{\eta}}) \subset L^2(\alpha)$. In particular, therefore, $A^{2,n}(V_{\boldsymbol{\eta}})$ is subnormal. This important property was used in the proof of Theorem 5.4 in Section 5.

We first consider the holomorphic functions on the one dimensional variety $V_{\boldsymbol{\eta}}$ as given in [7]. By definition, a function f is holomorphic in an analytic variety V of a complex manifold M if for every $x \in V$, there is a neighborhood $U_x \subset M$ and a holomorphic function F on U_x such that $F|_{V\cap U_x} = f|_{V\cap U_x}$. In [7], Clark proved that a holomorphic function f on $V_{\boldsymbol{\eta}}$ can be recovered in following sense: Define h on $\mathbb{D}^{n+1}$ by

$$h(z_1, \dots, z_n; \lambda) = \sum_{k=1}^{m} f(b_k(\lambda)) \prod_{j=1}^{n} \frac{(\lambda - \eta_j(z_j))}{((b_k(\lambda))_j - z_j)\eta_j'((b_k(\lambda))_j)},$$

where $b_k(\lambda)$, $k = 1, \dots, m$, is a pre-image of $\lambda \in \mathbb{D}$ under the map

$$\phi : V_{\underline{\eta}} \to \overline{\mathbb{D}}, \ \ \phi(z_1, \dots, z_n) = \eta_1(z_1),$$

and $(b_k(\lambda))_j$ is the j-th coordinate of $b_k(\lambda)$. Then (see Clark, Lemma 2.1 in [7]), as a function of λ, $h(z_1, \dots, z_n; \lambda)$ extends to a holomorphic $\mathcal{Q}_{\eta_1} \otimes \dots \otimes \mathcal{Q}_{\eta_n}$-valued function in $\mathbb{D}$, and

$$h(z_1, \dots, z_n; \eta_1(z_1)) = f(z_1, \dots, z_n),$$

for all $(z_1, \dots, z_n) \in V$.

We now proceed to construct the Hilbert space $A^{2,n}(V_{\boldsymbol{\eta}})$ of holomorphic functions on $V_{\boldsymbol{\eta}}$ (see Clark [7]). Let f be a holomorphic function on $V_{\boldsymbol{\eta}}$. Then

$$f(z_1, \dots, z_n) = \sum_{\nu=0}^{\infty} f_\nu(z_1, \dots, z_n)(\eta_1(z_1))^\nu,$$

where $f_\nu \in \mathcal{Q}_{\eta_1} \otimes \ldots \otimes \mathcal{Q}_{\eta_n}$. For each $0 < r < 1$, set

$$f_{[r]}(z_1, \ldots, z_n) := \sum_{\nu=0}^{\infty} f_\nu(z_1, \ldots, z_n) r^\nu (\eta_1(z_1))^\nu.$$

Now by the definition of h, we have

$$f_{[r]}(z_1, \ldots, z_n) = \sum_{k=1}^{m} f(b_k(r\eta_1(z_1))) \prod_{j=1}^{n} \frac{r\eta_1(z_1) - \eta_j(z_j)}{((b_k(r\eta_1(z_1)))_j - z_j)\eta_1'((b_k(r\eta_1(z_1)))_j)},$$

for $(z_1, \ldots, z_n) \in \partial V_{\boldsymbol{\eta}}$. Let $A^{2,n}(V_{\boldsymbol{\eta}})$ denote the Hilbert space of all holomorphic functions f on $V_{\boldsymbol{\eta}}$ such that

$$||f||_{2,n}^2 = 2(n+1) \int_0^1 ||f_{[r]}(z_1, \ldots, z_n)||_2^2 (1-r^2)^n r\, dr < \infty,$$

where

$$||f_{[r]}||_2 = \sup_{0 \le s < 1} \left(\int_0^{2\pi} \int |f_{[rs]}(w_1, \ldots, w_n)|^2 d\mu_{e^{i\theta}}(w) d\theta \right)^{1/2}.$$

Here the measure $d\mu_{e^{i\theta}}$ is a measure supported in the fibre over the point $e^{i\theta}$ under the map ϕ, that is,

$$\mu_{e^{i\theta}}(b_k(e^{i\theta})) = |\eta_1'((b_k(e^{i\theta}))_1) \ldots \eta_n'((b_k(e^{i\theta}))_n)|^{-1}.$$

We can represent the norm of f as

$$||f||_{2,n}^2 = 2(n+1) \int_0^1 \int_0^{2\pi} \int |f_{[r]}(w_1, \ldots, w_n)|^2 r(1-r^2)^n d\mu_{e^{i\theta}}(w) d\theta dr.$$

It follows from [7, Lemma 4.1] that the constant function 1 is in $A^{2,n}(V_{\boldsymbol{\eta}})$.

We can now define a measure μ on $V_{\boldsymbol{\eta}}$ as

$$\mu(E) = 2(n+1) \int_0^1 \int_0^{2\pi} \int (\chi_E)_{[r]}(w_1, \ldots, w_n) r(1-r^2)^n d\mu_{e^{i\theta}}(w) d\theta dr.$$

where χ_E is the characteristic function of the Borel set $E \subset V_{\boldsymbol{\eta}}$. With this measure we obtain that $A^{2,n}(V_{\boldsymbol{\eta}}) \subset L^2(\alpha)$.

Acknowledgment: The first two authors are grateful to Indian Statistical Institute, Bangalore Center for warm hospitality. The first named author's research work is supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094. The second namedauthor is supported by an INSPIRE faculty fellowship (IFA-MA-02) funded by DST. The third author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014.

References

[1] W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141-224.
[2] W. Arveson, *Subalgebras of C^*-algebras II*, Acta Math. **128** (1972), 271-308.
[3] L. Brown, R. Douglas and P. Fillmore, *Extension of C^*-algebras and K-homology*, Ann. of Math. **105** (1977), 265-324.
[4] A. Chattopadhyay, B. K. Das and J. Sarkar, *Tensor product of quotient Hilbert modules*, J. Math. Anal. Appl. **429** (2015), 727–747.
[5] A. Chattopadhyay, B. K. Das and J. Sarkar, *Star-generating vectors of Rudin's quotient modules*, J. Funct. Anal. **267** (2014), 4341–4360.
[6] X. Chen and K. Guo, *Analytic Hilbert modules*, π-Chapman & Hall/CRC Res. Notes Math. 433, 2003.
[7] D. Clark, *Restrictions of H^p functions in the polydisk*, Amer. J. math. **110** (1988), 1119-1152.
[8] B. K. Das and J. Sarkar, *Rudin's Submodules of $H^2(\mathbb{D}^2)$*, C. R. Acad. Sci. Paris **353** (2015), 51–55.
[9] K. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke Math. J. 164 (2015),2989-3004.
[10] R. G. Douglas and V. I. Paulsen, *Hilbert Modules over Function Algebras*, Research Notes in Mathematics Series, 47, Longman, Harlow, 1989.
[11] K. Guo and K. Wang, *Essentially normal Hilbert modules and K-homology*, Math. Ann. **340** (2008), 907-934.
[12] K. Guo and K. Wang, *Beurling type quotient modules over the bidisk and boundary representations*, J. Funct. Anal. **257** (2009), 3218-3238.
[13] K. Guo and P. Wang, *Essentially normal Hilbert modules and K-homology III: Homogeneous quotient modules on the bidisk*, Sci. China Ser. A **50** (2007), 387-411.
[14] W. He, *Boundary representations on co-invariant subspaces of Bergman space*, Proc. Amer. Math. Soc. **138** (2010), 615-622.
[15] K. Izuchi, T. Nakazi and M. Seto, *Backward shift invariant subspaces in the bidisc II*, J. Oper. Theory **51** (2004), 361-376.
[16] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.
[17] J. Sarkar, *Jordan blocks of $H^2(\mathbb{D}^n)$*, J. Oper. Theory **72** (2014), 371-385.
[18] J. Sarkar, A. Sasane and B. Wick, *Doubly commuting submodules of the Hardy module over polydiscs*, Studia Mathematica **217** (2013), 179-192.
[19] P. Wang, *The essential normality of $N_{\underline{\eta}}$-type quotient module of Hardy module on the polydisc*, Proc. Amer. Math. Soc. **142** (2014), 151-156.
[20] K. Zhu, *Restriction of the Bergman shift to an invariant subspace*, Quart. J. Math. Oxford Ser. (2) **48** (1997), 519-532.

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, India- 400076

E-mail address: `dasb@math.iitb.ac.in, bata436@gmail.com`

Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur 741 246, West Bengal, India

E-mail address: `sushil.gorai@iiserkol.ac.in`

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India

E-mail address: `jay@isibang.ac.in, jaydeb@gmail.com`